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Exercise Sheet 2 for „Einführung in das Rechnergestützte Arbeiten: Maple“

If you want to use the same environment as the on in the lecture, start Maple and choose „Worksheet mode“. Additionally switch „Input display:“ under „Tools“ → „Options“ in the „Display“ tab to „Maple Notation“ and apply those changes by clicking on „Apply globally“.
In the following parts, `expr` is used to mark Maple expressions.
For maple commands help is available by prepending the command with a question mark. `?int` for example opens the help page for the integration command.

1D-Plots, Derivatives, Integrals and Analytic Transformation of Expressions

Maple is able to perform some of the following transformations automatically with `simplify`, but sadly not all. In this exercise you will learn how Maple can do analytic tranformations with some aid of the user.

1 Solve **Task I** by using the commands `plot`, `diff` and `int`. Many obvious transformations before and after taking the derivative or integrating can be done with `simplify`.
For subtask **B**iv) using `combine(%,trig)` after taking the derivative is helpful.

Task I

A) Plot the following functions:

$$\text{(i)}\quad f(x) = \sinh(x) \qquad \text{(ii)}\quad f(x) = \ln|x| \qquad \text{(iii)}\quad f(x) = \frac{1}{1-x}$$

B) Calculate:

$$\text{(i)}\quad \frac{\mathrm{d}}{\mathrm{d}x} e^{n \ln x} \qquad \text{(ii)}\quad \frac{\mathrm{d}}{\mathrm{d}a} a^{x^2} \qquad \text{(iii)}\quad \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\tan\theta \cdot \cos\theta\right) \qquad \text{(iv)}\quad \frac{\mathrm{d}}{\mathrm{d}a}\left(\sin^2\left(x^2+a\right)\right)$$

C) Calculate the following integrals:

$$\text{(i)}\quad \int \mathrm{d}x \sin(ax) \qquad \text{(ii)}\quad \int \mathrm{d}a \sqrt{a+3} \qquad \text{(iii)}\quad \int_3^6 \frac{\mathrm{d}x}{x} \qquad \text{(iv)}\quad \int_0^4 \frac{x\mathrm{d}x}{\sqrt{x^2+9}}$$

2 Solve task **Task II**.
Calculate the antiderivative by using `int`. Ignore the constraints on the parameters a, b and c for now.
After that, use `assume` to tell Maple those constraints and use `int` again. What changed in the result?[1]

Task II Substitution Rule
Show by integrating that

$$\int \frac{\mathrm{d}x}{\sqrt{ax^2+2bx+c}} = -\frac{1}{\sqrt{-a}} \arcsin \frac{ax+b}{\sqrt{b^2-ac}}, \qquad \text{falls} \quad a<0 \quad \text{and} \quad b^2-ac>0.$$

[1] The two results are of course the same when considering the version of the contained functions for complex valued arguments.

Analytic solutions of ODEs, Limits, Series expansions

3 Solve **Task III**.

a) Use `dsolve`, to solve the differential equation (**??**) for the velocity components $v_x(t), v_z(t)$:[2].

$$\frac{\mathrm{d}}{\mathrm{d}t} v_x(t) = -k \cdot v_x(t) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} v_z(t) = -k \cdot v_z(t) - g \tag{1}$$

with the boundary condition $v_x(0) = v_0 \cos\theta$ and $v_z(0) = v_0 \sin\theta$ respectively.

b) Use `dsolve`, to solve the two differential equations for $r_x(t)$ and $r_z(t)$.

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} r_x(t) = -k \cdot \frac{\mathrm{d}}{\mathrm{d}t} r_x(t) \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} r_z(t) = -k \cdot \frac{\mathrm{d}}{\mathrm{d}t} r_z(t) - g \tag{2}$$

with the boundary condition

$$r_x(0) = 0, \quad \frac{\mathrm{d}r_x}{\mathrm{d}t}(0) = v_0 \cos\theta \qquad \text{and} \qquad r_z(0) = 0, \quad \frac{\mathrm{d}r_z}{\mathrm{d}t}(0) = v_0 \sin\theta \tag{3}$$

respectively. Then extract the solutions using `rhs` or as was demonstrated in the lecture with the help of `subs` for these two components of $\vec{r}(t)$.

Find out using `?plot, parametric`, how to plot trajectories, and plot these for the parameter values $g = 9.81$, $k = 0.1$, $v_0 = 100$ and $\theta = \pi/6$.

Now directly solve the differential equations numerically and plot the solution with `odeplot`.

c) Maple can calculate the Limit with `limit`, if $k > 0$ if set. This can be achieved conveniently with the `assuming` directive, which sets this property of k only locally for this single command, instead of globally for all calculations.

d*) Show the relation below by taking the limit $k \to 0$. Calculate the first correction term using series expansion around $k = 0$ (with `series`).

Task III Projectile motion/Ballistic trajectory

A point mass is thrown upwards at an angle θ relative to the horizontal orientation. At time $t = 0$ the point mass is at position $\vec{r}(0) = (0,0,0)^T$ and the absolute value of the velocity is $|\vec{v}(0)| = v_0$. Due to air drag the acceleration is given by $\vec{a}(t) = -k\vec{v}(t) - g\vec{e}_z$.

Through integration of the equation

$$\vec{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{v}(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{r}(t) \tag{4}$$

you get $\vec{v}(t)$ and subsequently $\vec{r}(t)$. The respective integration constants fixed by the condition $\vec{v}(0) = v_0(\cos\theta, 0, \sin\theta)^T$ and $\vec{r}(0)$.

A) Determine the velocity of the point mass as a function of time.

B) Determine the trajectory $\vec{r}(t)$ of the point mass and sketch it.

C) Show that in the limit $t \to \infty$ the point mass reaches a finite terminal velocity and calculate its value.

D) Show for small air drag ($k \to 0$) the height is approximately given by

$$r_z(t) = v_z(0)t - \frac{gt^2}{2}$$

.

[2] There should be one differential equation for each component. In this exercise no vectors shall be used.

3D Plots of Trajectories, Norm of Vectors

[4] Solve **Task IV** with Maple. Consider thereby the following notes:

a) Define the given trajectory as a `Vector` of expressions. Using `eval` set the involved parameters to reasonable values save the result in a second vector. You can now plot it using `spacecurve` from the package `plots` (call `with(plots)` beforehand!).

b) To calculate the time derivative of the vector Maple unfortunately need the additional package `VectorCalculus`.

c) `Norm(expr)` from the package `VectorCalculus` calculates the (euclidean) norm of a vector and therein implicitly assumes real vectors[3].
The package `LinearAlgebra` also provides the function `Norm`, where you have to specify with `Norm(expr,2)` explicitly the euclidean norm. Here you have to tell Maple with `assume` that all involved parameters are real and $R > 0$. Then `simplify` actually produces a simpler expression from the norm.

Task IV Particle in a magnetic field
An electrically charged particle moves in a magnetic field that is constant in space and time. Under the influence of the Lorentz force the particle moves along a trajectory that is given by the following parametrization:

$$x(t) = R\cos(\omega t), \qquad y(t) = R\sin(\omega t), \qquad z(t) = h\frac{\omega t}{2\pi}$$

A) Sketch the trajectory.
B) Specify the velocity and acceleration vectors.
C) Determine the norm of the velocity.

[3] i.e. for complex vectors the results will be wrong!

Linear Algebra, Matrix Calculations, Eigenvalues, Eigenvectors, Linear Systems

5* Solve **Task V** with the `LinearAlgebra` package. Take the following hints into account:

a) A matrix vector product can be calculated with `.`, e.g. `C:=A.B;`. With `Transpose` row vectors can be transposed into column vectors.

b) The 3×3 identity matrix is available with `Matrix(3,3,shape=identity)` (as are many other special matrices).

c) A determinant can be calcuated with `Determinant`. Compare the polynomial with the result of `CharacteristicPolyno` of A.

d) Obtain the roots of the characteristic polynomial with `solve` and compare the them with the eigenvalues of A with can be calculated with `Eigenvalues`.

Task V Matrix Calculations

A) Calculate the products AB and BA:

$$\text{(i)}\quad A = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}; \qquad \text{(ii)}\quad A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

Is the matrix product commutative ($AB - BA = 0$)?

B) Calcualte $B = A - \lambda \mathbb{1}$ with

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and the identity matrix} \quad \mathbb{1}$$

C) Calculate the determinant of B.

D) Calculate the three roots λ of $\det B = 0$.